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Cartesian products over extended index matrices

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Abstract: By the moment, two different Cartesian products and defined over intuitionistic fuzzy index matrices. In the preset paper, seven new definitions of operation Cartesian product are introduced in the more general case of extended index matrices and some of their properties are studied. In a particular case, it is shown how these definitions can be modified for the intuitionistic fuzzy index matrices.

Keywords: Cartesian product, Index matrix, Intuitionistic fuzzy index matrix, Intuitionistic fuzziness.

AMS Classification: 03E72, 11C20.

1 Introduction

Here, as a continuation of the development of the Index Matrix (IM) theory [1, 3, 10], and esspecially of [9], in Section 3, we discuss seven new Cartesian type of products over Extended IMs (EIMs).

In [10], a new operation called "concatenation" and denoted by $\textcircled{\wedge}$, was introduced over standard IM. In [9], it was modified to the form of the two types of Cartesian products, described in Section 2.

2 Basic definition

Firstly, following [3], the definition of an EIM is proposed.

Let $\mathcal{I}$ be a fixed set of indices,

$$\mathcal{I}^n = \{\langle i_1, i_2, ..., i_n\rangle | (\forall j : 1 \le j \le n)(i_j \in \mathcal{I})\}$$

and

$$\mathcal{I}^* = \bigcup_{1 \le n < \infty} \mathcal{I}^n.$$

In the present research, $n = 1$ or 2.

Let everywhere below $\mathcal{X}$ be a fixed set of some objects. In particular cases, they can be either real numbers, or just numbers 0 or 1; logical variables, propositions or predicates, etc.

Let operations $\circ, * : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be given.

We call the object $[K, L, \{a_{k_i,l_j}\}]$ with index sets K and L $(K, L \subset \mathcal{I}^*)$, an EIM. It has the form

$$[K, L, \{a_{k_i,l_j}\}] \equiv \begin{array}{c|ccccc} & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & a_{k_1,l_1} & \dots & a_{k_1,l_j} & \dots & a_{k_1,l_n} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_i & a_{k_i,l_1} & \dots & a_{k_i,l_j} & \dots & a_{k_i,l_n} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_m & a_{k_m,l_1} & \dots & a_{k_m,l_j} & \dots & a_{k_m,l_n} \end{array},$$

where $K = \{k_1, k_2, ..., k_m\}$, $L = \{l_1, l_2, ..., l_n\}$, for $1 \le i \le m$, and $1 \le j \le n$: $a_{k_i,l_j} \in \mathcal{X}$.

Secondly, we give some remarks on Intuitionistic Fuzzy Logics (see, e.g., [4]) and especially, of their particular case, Intuitionistic Fuzzy Pairs (IFPs; see [8]). The IFP is an object in the form $\langle a, b\rangle$, where $a, b \in [0, 1]$ and $a + b \le 1$ which is used as an evaluation of some object or process and which components (a and b) are interpreted as degrees of membership and non-membership, or degrees of validity and non-validity, or degrees of correctness and non-correctness, etc.

The Intuitionistic Fuzzy IM (IFIM, see [3]) is defined by:

$$[K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle\}]$$

$$= \begin{array}{c|ccccc} & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1}\rangle & \dots & \langle \mu_{k_1,l_j}, \nu_{k_1,l_j}\rangle & \dots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n}\rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & \langle \mu_{k_i,l_1}, \nu_{k_i,l_1}\rangle & \dots & \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle & \dots & \langle \mu_{k_i,l_n}, \nu_{k_i,l_n}\rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1}\rangle & \dots & \langle \mu_{k_m,l_j}, \nu_{k_m,l_j}\rangle & \dots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n}\rangle \end{array},$$

where for every $1 \le i \le m, 1 \le j \le n$: $a_{k_i,l_j} = \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle$ and $\mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \in [0, 1]$.

Let us have two IFIMs $A = [K, L, \{a_{k_i,l_j}\}]$ and $B = [P, Q, \{b_{p_r,q_s}\}]$, where a_{k_i,l_j} and b_{p_r,q_s} are IFPs or real numbers.

The first type of Cartesian product is the following (see [9]):

$$A \times_C B = [K \times P, L \times Q, \{c_{\langle k_i, p_r\rangle, \langle l_j, q_s\rangle}\}],$$

where

$$c_{\langle\langle k_i, p_r\rangle, \langle l_j, q_s\rangle\rangle} = \langle a_{k_i, l_j}, b_{p_r, q_s}\rangle$$

and operation $\times$ between K and P, and between L and Q is the standard set-theoretical Cartesian product.

The second type of Cartesian product is the following (see [9]):

$$A \times_{\circ,*} B = [K \times P, L \times Q, \{c_{\langle k_i, p_r\rangle, \langle l_j, q_s\rangle}\}],$$

where

$$c_{\langle\langle k_i, p_r\rangle, \langle l_j, q_s\rangle\rangle} = (\circ, *)\langle a_{k_i, l_j}, b_{p_r, q_s}\rangle,$$

and for the suitable variables t, u, v, w, in some cases (e.g., conjunction or disjunction)

$$(\circ, *)\langle\langle t, u\rangle, \langle v, w\rangle\rangle = \langle \circ(t, v), *(u, w)\rangle$$

and in others (e.g., implication)

$$(\circ, *)\langle\langle t, u\rangle, \langle v, w\rangle\rangle = \langle \circ(u, v), *(t, w)\rangle$$

with respect to the type of the operation that the pair $(\circ, *)$ represents.

3 Main results

An n-Dimensional EIM (n-DEIM), with index sets $K_1, K_2, ..., K_n$ ($K_1, K_2, ..., K_n \subseteq \mathcal{I}^*$) and elements from the set $\mathcal{X}$, is called the object:

$$A = [K_1, K_2, ..., K_n, \{a_{k_{1,s_1}, k_{2,s_2}, ..., k_{n,s_n}}\}]$$

where $K_i = \{k_{i,1}, k_{i,2}, ..., k_{i,m_i}\}$, $m_i \geq 1$ and $a_{k_{1,s_1}, k_{2,s_2}, ..., k_{n,s_n}} \in \mathcal{X}$ for $1 \leq i \leq n$ and $1 \leq s_i \leq m_i$ (see [5]).

Here, we suppose that if $a, b \in \mathcal{X}$, then $\langle a, b\rangle \in \mathcal{X}$.

The n-DEIM A has the following $\frac{n(n-1)}{2}$ different representations as 2-DEIM:

$$A = \begin{array}{c|ccc}
\langle k_{3,s_3}, ..., k_{n,s_n}\rangle & k_{2,1} & \dots & k_{2,m_2} \\ \hline
k_{1,1} & a_{k_{1,1}, k_{2,1}, k_{3,s_3}, ..., k_{n,s_n}} & \dots & a_{k_{1,1}, k_{2,m_2}, k_{3,s_3}, ..., k_{n,s_n}} \\
\vdots & \vdots & \ddots & \vdots \\
k_{1,i} & a_{k_{1,i}, k_{2,1}, k_{3,s_3}, ..., k_{n,s_n}} & \dots & a_{k_{1,i}, k_{2,m_2}, k_{3,s_3}, ..., k_{n,s_n}} \\
\vdots & \vdots & \ddots & \vdots \\
k_{1,m_1} & a_{k_{1,m_1}, k_{2,1}, k_{3,s_3}, ..., k_{n,s_n}} & \dots & a_{k_{1,m_1}, k_{2,m_2}, k_{3,s_3}, ..., k_{n,s_n}}
\end{array}$$

$$= \dots =$$

$\langle k_{1,s_1},...,k_{n-2,s_{n-2}}\rangle$	$k_{n,1}$	$\dots$	k_{n,m_n}
$k_{n-1,1}$	$a_{k_{1,s_1},...,k_{n-2,s_{n-2}},k_{n-1,1},k_{n,1}}$	$\dots$	$a_{k_{1,s_1},...,k_{n-6,s_{n-6}},k_{n-1,1},k_{n,m_n}}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$k_{n-1,i}$	$a_{k_{1,s_1},...,k_{n-6,s_{n-6}},k_{n-1,i},k_{n,1}}$	$\dots$	$a_{k_{1,s_1},...,k_{n-6,s_{n-6}},k_{n-1,i},k_{n,nm}}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$k_{n-1,m_{n-1}}$	$a_{k_{1,s_1},...,k_{n-6,s_{n-6}},k_{n-1,m_{n-1}},k_{n,1}}$	$\dots$	$a_{k_{1,s_1},...,k_{n-2,s_{n-2}},k_{n-1,m_{n-1}},k_{n,nm}}$

Let us have two n-DEIMs

$$A = [K_1, K_2, ..., K_n, \{a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}\}]$$

and

$$B = [L_1, L_2, ..., L_n, \{b_{l_{1,t_1},l_{2,t_2},...,l_{n,t_n}}\}],$$

and let $X \times Y$ be the standard set-theoretical Cartesian product.

First, we introduce the Cartesian product

$$A \times_C B = [K_1 \times L_1, K_2 \times L_2, ..., K_n \times L_n, \{\langle a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}, b_{l_{1,t_1},l_{2,t_2},...,l_{n,t_n}}\rangle\}].$$

When $n = 2$, we obtain the first Cartesian product from [9] for the case of two IFIMs.

Let for the three elements $a, b, c \in \mathcal{X}$ the equalities

$$\langle\langle a, b\rangle, c\rangle = \langle a, b, c\rangle = \langle a, \langle b, c\rangle\rangle$$

be valid.

Theorem 1. The operation $\times_C$ is associative, but not commutative.

Proof. Let us have three n-DEIMs A, B, C so that A and B are the n-DEIMs from above and

$$C = [M_1, M_2, ..., M_n, \{c_{m_{1,u_1},m_{2,u_2},...,m_{n,u_n}}\}].$$

Then

$$\begin{aligned}
&(A \times_C B) \times_C C \\
&= [K_1 \times L_1, K_2 \times L_2, ..., K_n \times L_n, \{\langle a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}, b_{l_{1,t_1},l_{2,t_2},...,l_{n,t_n}}\rangle\}] \times_C C \\
&= [(K_1 \times L_1) \times M_1, (K_2 \times L_2) \times M_2), ..., (K_n \times L_n) \times M_n, \\
&\{\langle\langle a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}, b_{l_{1,t_1},l_{2,t_2},...,l_{n,t_n}}\rangle, c_{m_{1,u_1},m_{2,u_2},...,m_{n,u_n}}\rangle\}] \\
&= [K_1 \times L_1 \times M_1, K_2 \times L_2 \times M_2, ..., K_n \times L_n \times M_n, \\
&\{\langle a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}, b_{l_{1,t_1},l_{2,t_2},...,l_{n,t_n}}, c_{m_{1,u_1},m_{2,u_2},...,m_{n,u_n}}\rangle\}] \\
&= [K_1 \times (L_1 \times M_1), K_2 \times (L_2 \times M_2), ..., K_n \times (L_n \times M_n), \\
&\{\langle a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}, \langle b_{l_{1,t_1},l_{2,t_2},...,l_{n,t_n}}, c_{m_{1,u_1},m_{2,u_2},...,m_{n,u_n}}\rangle\rangle\}] \\
&= A \times [L_1 \times M_1, L_2 \times M_2, ..., L_n \times M_n, \{\langle b_{l_{1,t_1},l_{2,t_2},...,l_{n,t_n}}, c_{m_{1,u_1},m_{2,u_2},...,m_{n,u_n}}\rangle\}] \\
&A \times_C (B \times_C C).
\end{aligned}$$

From the definition of operation $\times_C$ it is clear that it is not commutative, because from set-theoretical point of view, for example, $K_1 \times K_2 \neq K_2 \times K_1$ in the general case, and the same is valid for L-index sets.

The operation $\times_C$ can be modified for two EIM so that the first one is an m-DEIM and the second one - n-DEIM. Let

$$A = [K_1, K_2, ..., K_m, \{a_{k_{1,s_1},k_{2,s_2},...,k_{m,s_m}}\}]$$

and B has the above form. Then

$$A \times^C B = [K_1, K_2, ..., K_m, L_1, L_2, ..., L_n, \{\langle a_{k_{1,s_1},k_{2,s_2},...,k_{m,s_m}}, b_{l_{1,t_1},l_{2,t_2},...,l_{n,t_n}}\rangle\}].$$

By the same manner as above, it is proved

Theorem 2. The operation $\times^C$ is associative, but not commutative.

The following two Cartesian products are extensions of the two above operations.

$$A \times_{C,\circ} B = [K_1 \times L_1, K_2 \times L_2, ..., K_n \times L_n, \{a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}} \circ b_{l_{1,t_1},l_{2,t_2},...,l_{n,t_n}}\}].$$

$$A \times_\circ^C B = [K_1, K_2, ..., K_m, L_1, L_2, ..., L_n, \{a_{k_{1,s_1},k_{2,s_2},...,k_{m,s_m}} \circ b_{l_{1,t_1},l_{2,t_2},...,l_{n,t_n}}\}].$$

Theorem 3. The operations $\times_{C,\circ}$ and $\times_\circ^C$ are associative, but not commutative.

Let for every three objects $a, b, c \in \mathcal{X}$:

$$a \circ (b * c) = (a \circ b) * (a \circ c)$$

and/or

$$(a * b) \circ c = (a \circ c) * (b \circ c)$$

Theorem 4. For every three EIM A, B, C:

$$A \times_\circ^C (B \times_{C,*} C) = (A \times_\circ^C B) \times_{C,*} (A \times_\circ^C C)$$

and/or

$$(A \times_*^C B) \times_\circ^C C = (A \times_\circ^C C) \times_{C,*} (B \times_\circ^C C),$$

$$A \times_{C,\circ} (B \times_{C,*} C) = (A \times_{C,\circ} B) \times_{C,*} (A \times_{C,\circ} C)$$

and/or

$$(A \times_{C,*} B) \times_{C,\circ} C = (A \times_{C,\circ} C) \times_{C,*} (B \times_{C,\circ} C).$$

Now, we introduce a new object, that can be interpreted as a result of the operation Cartesian product.

Let the following m in number multi-DEIMs be given for $i = 1, 2, \ldots, m$, i.e., the different EIMs can have different dimensions, i.e., $n_1, n_2, \ldots, n_m$:

$$A_i = [K; L_1^i, L_2^i, \ldots, L_{s_i}^i \{a_{k,l_1^i,\ldots,l_{r_i}^i}\}]$$

so that the index set K is equal for all of them, while the sets $L_1^1, L_2^1, \ldots, L_{s_m}^m$ do not have equal elements. Now we define

$$\Delta(A_1, A_2, \ldots, A_m) = [K; L_1^1, \ldots, L_{s_m}^m, \{b_{k,l_1^1,l_2^1,\ldots,l_{r_m}^m}\}],$$

where for each $i : 1 \leq i \leq m$

$$b_{k,l_1^1,l_2^1,\ldots,l_1^i,\ldots,l_{r_i}^i,\ldots,l_{r_m}^m} = a_{k,l_1^i,\ldots,l_{r_i}^i}.$$

Geometrically, this object can be represented as it is shown on Fig. 1. By this reason we can call it an IM-Book (IMB). This object can be extended in two directions.

First direction is obtained when we check the index set K of the EIMs $A_1, A_2, \ldots, A_m$ with a set (equal for each EIMs) $K_1, K_2, \ldots, K_k$, while the second direction is to omit the condition that sets $L_1^1, L_2^1, \ldots, L_{s_m}^m$ do not have equal elements, changing it with the possibility the sets L_p^i and L_q^i to have joint elements for $p \neq q$, but this will be not valid for the sets L_r^p and L_r^q for $p \neq q$. In this case, the result of applying of operator Δ over EIMs $A_1, A_2, \ldots, A_m$ has two forms:

$$\Delta_{\cup}(A_1, A_2, \ldots, A_m) = [K; \bigcup_{i=1}^{m} L_1^i, \bigcup_{i=1}^{m} L_2^i, \ldots, \bigcup_{i=1}^{m} L_{s_i}^i \{b_{k,l_1^1,l_2^1,\ldots,l_{r_m}^m}\}]$$

and

$$\Delta_{\cap}(A_1, A_2, \ldots, A_m) = [K; \bigcap_{i=1}^{m} L_1^i, \bigcap_{i=1}^{m} L_2^i, \ldots, \bigcap_{i=1}^{m} L_{s_i}^i \{b_{k,l_1^1,l_2^1,\ldots,l_{r_m}^m}\}],$$

where some of the sets L_j^i can be the empty set.

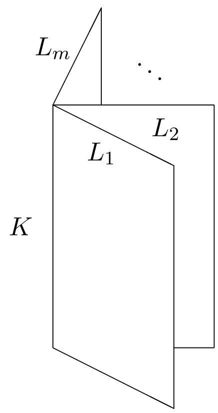

Fig. 1.

4 Conclusion

The so constructed new operations from Cartesian type over EIMs can be used for different aims. On the one hand, they can be used for formal description of Big Data-structures and on the other - for complex procedures for intercriteria analysis (see, e.g., [6, 7] of data. These two directions of applications will be discussed in the near future.

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